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Existence and uniqueness for a non-coercive lubrication problem

Guy Bayada [a,*], Mohamed El Alaoui Talibi [b], Khalid Ait Hadi [b]

[a] *CNRS UMR 5585-5514, Bât 401, INSA de Lyon Math., 69621 Villeurbanne cedex, France*
[b] *Département de Mathématiques, Faculté des Sciences Semlalia, BP 2390, Marrakech, Morocco*

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Abstract

The purpose of this paper is to study a mathematical model of hydrodynamic lubrication when cavitation takes place. The obtained equation is a non-linear degenerate partial differential equation.
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1. Introduction

The study of free boundary problems in the hydrodynamic lubrication gave place to many works covering some fundamental and applied aspects. For lubricated devices with Newtonian fluid, the classical thin film Reynolds equation is [14]:

$$\frac{\partial}{\partial x}\left(\frac{h^3}{12\eta}\frac{\partial p}{\partial x}\right)+\frac{\partial}{\partial y}\left(\frac{h^3}{12\eta}\frac{\partial p}{\partial y}\right)=\frac{\partial}{\partial x}\left(\frac{hU\rho}{2}\right) \tag{1.1}$$

where p is the pressure of the lubricant, $h(x)$ is the given film thickness, ρ is the density of lubricant, η is the viscosity of the lubricant, U is the relative velocity of the surfaces in which lubricant takes place and Ω is a given domain of the (x, y) plane.

* Corresponding author.
E-mail address: guy.bayada@insa-lyon.fr (G. Bayada).

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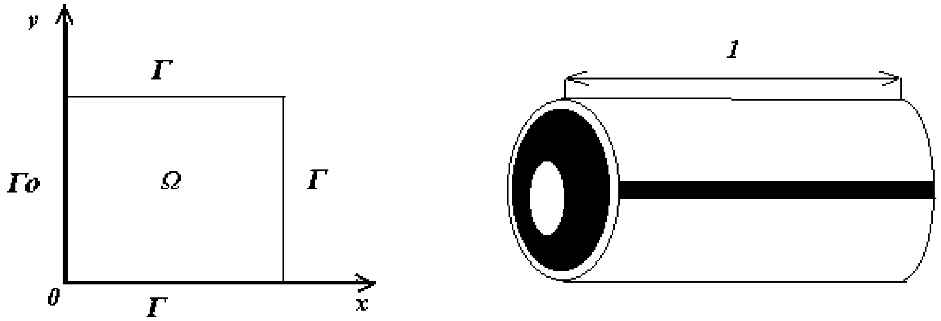

Fig. 1. Geometry of a cylindrical journal bearing.

Equation (1.1) is to be used only on an unknown part of Ω in which pressure is strictly greater than the vaporization pressure (often taken as zero). In the other part, namely cavitation region, Reynolds equation is no longer valid and a description of the saturation in that region is to be introduced [4]. Through the numerous equations describing the phenomena, we will focus in this paper in the one introduced by Elrod [12] and widely used for numerical purpose [18,19]. A more precise description of the modelization will appear in Section 4.1. From a mathematical point of view the problem is written in the following form which takes a compressibility operator B into account:

$$\begin{cases} \text{Find } u \in L^2(\Omega) \text{ with } u^+ \in H^1(\Omega) \text{ such that} \\ \dfrac{\partial}{\partial x}\left(\dfrac{h^3}{12\eta}\dfrac{\partial u^+}{\partial x}\right) + \dfrac{\partial}{\partial y}\left(\dfrac{h^3}{12\eta}\dfrac{\partial u^+}{\partial y}\right) = \dfrac{\partial}{\partial x}\left(\dfrac{hUB(u)}{2}\right), \end{cases} \tag{1.2}$$

where $B(.)$ is a known operator associated to a pressure compressibility relation:

$$B(.): \mathbb{R} \to \mathbb{R} \quad \text{such that} \quad 0 < B'(r) \leqslant \frac{1}{\beta}, \quad B(0) = 1, \tag{1.3}$$

for a positive constant β (for which we will give a physical meaning in Section 4) and $u^+ = \sup(u, 0)$. Equation (1.2) has to be supplemented by some boundary conditions depending of the physical device considered. It is possible to use various boundary conditions, see [8,12,13]. In the following, we will consider the particular case of a journal bearing (Fig. 1) so that:

$\Omega := [0,1]^2$ and an input flow $\theta_0(y)$ given on $\Gamma_0 = 0 \times [0,1]$ and the pressure is zero on $\partial\Omega \setminus \Gamma_0$.

The goal of this paper is to study existence and uniqueness properties for (1.2). The presence of the degenerate term u^+ in the left-hand side prevents us to use classical techniques. The paper is organized as follows. In Section 2, existence of a weak solution of (1.2) is proved by using a generalization of Ky Fan's lemma. Condition of existence so obtained covered a wider range of parameters than the one issued from the Schauder fixed point approach in [8] and devoted to the particular case compressibility pressure law: $B(u) = 1 + \frac{1}{\beta}u$ or the one issued from a 1-dimensional approach like in [1]. Section 3 is devoted to the proof of uniqueness of the weak solution of (1.2). So, monotonicity property with respect of the input flow parameter θ_0 is gained without any regularity assumption and uniqueness is obtained. At last, in Section 4 we give some physical aspect which motivates the introduction of (1.2) and we study a related asymptotic case both from theoretical and numerical aspects.

2. Existence of a solution

Let $h(x) \geqslant \alpha > 0$ be a Lipschitz continuous function and B an operator satisfying (1.3); we consider the following normalized problem:

$$(\mathcal{P}) \quad \begin{cases} \text{Find } u \in L^2(\Omega) \text{ verifying } u^+ \in V,\ B(u) \geqslant 0, \\ \displaystyle\int_\Omega h^3 \nabla u^+ \nabla \phi \, dx\, dy - \int_\Omega h B(u) \frac{\partial \phi}{\partial x} \, dx\, dy = \int_{\Gamma_0} \theta_0 h \phi \, dy \quad \forall \phi \in V, \\ \theta_0 \in L^\infty(\Gamma_0),\ 0 \leqslant \theta_0 \leqslant 1 \end{cases}$$

with $V := \{\varphi \in H^1(\Omega),\ \varphi_{/\Gamma} = 0\}$, $\Gamma_0 = \{(x, y) \in \partial\Omega,\ x = 0\}$; $\Gamma = \partial\Omega \backslash \Gamma_0$.

Remark 2.1. Taking into account that $H_0^1(\Omega) \subset V \subset H^1(\Omega)$ and that Γ is a subset of positive measure (for the surface measure on $\partial\Omega$) the extension of Poincaré's inequality [15]:

$$\|v\|_{L^2(\Omega)} \leqslant C_p \|\nabla v\|_{(L^2(\Omega))^2}, \tag{2.1}$$

holds for all v in V and $\|\varphi\|_V = \|\nabla\varphi\|_{(L^2(\Omega))^2}$ is a norm in V equivalent to the one induced by the usual norm in $H^1(\Omega)$.

2.1. Approached problems

Let us write $\nabla u^+ = g(u)\nabla u$ with

$$g(u) = \begin{cases} 1 & \text{if } u \geqslant 0, \\ 0 & \text{if } u < 0. \end{cases}$$

For $\epsilon > 0$, we introduce the continuous function $g_\epsilon(u)$ defined by

$$g_\epsilon(u) = \begin{cases} 1 & \text{if } u > 0, \\ -\left(1 - \frac{1}{\epsilon}\right)u + 1 & \text{if } -\epsilon \leqslant u \leqslant 0, \\ \epsilon & \text{if } u < -\epsilon, \end{cases}$$

and we consider the following problem:

$$(\mathcal{P}_\epsilon) \quad \begin{cases} \text{Let } \theta_0 \in L^\infty(\Gamma_0),\ 0 \leqslant \theta_0 \leqslant 1;\ \text{find } u^\epsilon \in V \text{ such that} \\ \displaystyle\int_\Omega h^3 g_\epsilon\big(u^\epsilon\big) \nabla u^\epsilon \nabla \phi \, dx\, dy - \int_\Omega h\big(B\big(u^\epsilon\big)\big)^+ \frac{\partial \phi}{\partial x} \, dx\, dy \\ \displaystyle\quad = \int_{\Gamma_0} \theta_0 h \phi \, dy \quad \forall \phi \in V. \end{cases} \tag{2.2}$$

Lemma 2.2. *If u^ϵ is a solution of $(\mathcal{P}_\epsilon)$ then $B(u^\epsilon) \geqslant 0$.*

Proof. We take in (2.2) $\phi = (B(u^\epsilon))^-$ with u^ϵ being a solution of $(\mathcal{P}_\epsilon)$. We get

$$\int_{\{B(u^\epsilon) \leqslant 0\}} h^3 g_\epsilon\big(u^\epsilon\big) B'\big(u^\epsilon\big) \big(\nabla u^\epsilon\big)^2 \, dx\, dy + \int_{\Gamma_0} \theta_0 h \big(B\big(u^\epsilon\big)\big)^- \, dy = 0,$$

and since $\int_{\Gamma_0} \theta_0 h(B(u^\epsilon))^- \, dy \geqslant 0$ then

$$\int\limits_{\{B(u^\epsilon)\leqslant 0\}} h^3 g_\epsilon\big(u^\epsilon\big) B'\big(u^\epsilon\big)\big(\nabla u^\epsilon\big)^2 \, dx\, dy = 0,$$

i.e., $\nabla(B(u^\epsilon))^- = 0$. It follows that $(B(u^\epsilon))^- = 0$ and $B(u^\epsilon) \geqslant 0$ since $B(u^\epsilon) = 1$ on Γ. □

Remark 2.3. Equation (2.2) is equivalent to

$$\int\limits_{\Omega} h^3 g_\epsilon\big(u^\epsilon\big)\nabla u^\epsilon \nabla\phi \, dx\, dy - \int\limits_{\Omega} hH\big(B\big(u^\epsilon\big)\big)B\big(u^\epsilon\big)\frac{\partial\phi}{\partial x}\, dx\, dy$$
$$= \int\limits_{\Gamma_0} \theta_0 h\phi \, dy \quad \forall\phi \in V,$$

with $H(.)$ being the Heaviside graph,

$$H(p) = \begin{cases} 0 & \text{if } p < 0, \\ 1 & \text{if } p \geqslant 0. \end{cases}$$

Let us regularize $H(.)$ by using the penalized function H_η such that

$$H_\eta(u) = \begin{cases} 0 & \text{if } u < 0, \\ \frac{u}{\eta} & \text{if } 0 \leqslant u \leqslant \eta, \\ 1 & \text{if } u > \eta, \end{cases}$$

and let us consider the following problem:

$$(\mathcal{P}_{\eta\epsilon}) \quad \begin{cases} \text{Let } \theta_0 \in L^\infty(\Gamma_0),\ 0 \leqslant \theta_0 \leqslant 1;\ \text{find } u^{\eta\epsilon} \in V \text{ such that} \\ \displaystyle\int\limits_{\Omega} h^3 g_\epsilon\big(u^{\eta\epsilon}\big)\nabla u^{\eta\epsilon}\nabla\phi \, dx\, dy - \int\limits_{\Omega} hH_\eta\big(B\big(u^{\eta\epsilon}\big)\big)B\big(u^{\eta\epsilon}\big)\frac{\partial\phi}{\partial x}\, dx\, dy \\ \quad \displaystyle = \int\limits_{\Gamma_0} \theta_0 h\phi \, dy \quad \forall\phi \in V. \end{cases} \tag{2.3}$$

Let G_ϵ be defined as follows: $G_\epsilon(u) = \int_0^u g_\epsilon(z)\, dz$, G_ϵ is continuous, increasing so that a reciprocal function G_ϵ^{-1} exists.

Then Eq. (2.3) is rewritten as follows:

$$\int\limits_{\Omega} h^3 \nabla G_\epsilon\big(u^{\eta\epsilon}\big)\nabla\phi \, dx\, dy - \int\limits_{\Omega} hH_\eta\big(B\big(u^{\eta\epsilon}\big)\big)B\big(u^{\eta\epsilon}\big)\frac{\partial\phi}{\partial x}\, dx\, dy$$
$$= \int\limits_{\Gamma_0} \theta_0 h\phi \, dy \quad \forall\phi \in V.$$

Let us introduce $v^{\eta\epsilon} = G_\epsilon(u^{\eta\epsilon})$. We consider the new problem:

$$(\mathcal{Q}_{\eta\epsilon})\quad \begin{cases} \text{Let } \theta_0 \in L^\infty(\Gamma_0),\ 0 \leqslant \theta_0 \leqslant 1;\ \text{find } v^{\eta\epsilon} \in V \text{ such that} \\ \displaystyle\int_\Omega h^3 \nabla v^{\eta\epsilon} \nabla\phi \, dx\, dy - \int_\Omega h H_\eta\big(B\big(G_\epsilon^{-1}\big(v^{\eta\epsilon}\big)\big)\big) B\big(G_\epsilon^{-1}\big(v^{\eta\epsilon}\big)\big) \frac{\partial\phi}{\partial x}\, dx\, dy \\ \displaystyle\qquad = \int_{\Gamma_0} \theta_0 h \phi \, dy \quad \forall\phi \in V. \end{cases} \tag{2.4}$$

2.2. *Existence of a solution for the problem* $(\mathcal{Q}_{\eta\epsilon})$

Let us consider the technical lemma:

Lemma 2.4. *Let C_p be the imbedding constant introduced in* (2.1). *If $h(.)$ satisfies*

$$\min_{\{x\in[0,1]\}} \big(h^3(x)\big) - \frac{1}{\beta} C_p \|h\|_{L^\infty(\Omega)} > 0 \tag{2.5}$$

then for all solutions $w_{\eta\epsilon}$ in V of the following inequality:

$$\int_\Omega h^3 \big(\nabla w^{\eta\epsilon}\big)^2 \, dx\, dy - \int_\Omega h H_\eta\big(B\big(G_\epsilon^{-1}\big(w^{\eta\epsilon}\big)\big)\big) B\big(G_\epsilon^{-1}\big(w^{\eta\epsilon}\big)\big) \frac{\partial w^{\eta\epsilon}}{\partial x}\, dx\, dy \\ \leqslant \int_{\Gamma_0} \theta_0 h w^{\eta\epsilon} \, dy, \tag{2.6}$$

the following estimate holds (*where C_1 is independent of ϵ and η*):

$$\big\|w^{\eta\epsilon}\big\|_V \leqslant C_1.$$

Proof. From (2.6) we have

$$\int_\Omega h^3 \big(\nabla w^{\eta\epsilon}\big)^2 \, dx\, dy \\ \leqslant \int_{\{B(G_\epsilon^{-1}(w^{\eta\epsilon}))\geqslant\eta\}} h B\big(G_\epsilon^{-1}\big(w^{\eta\epsilon}\big)\big) \frac{\partial w^{\eta\epsilon}}{\partial x}\, dx\, dy \\ + \int_{\{0\leqslant B(G_\epsilon^{-1}(w^{\eta\epsilon}))<\eta\}} \frac{h(B(G_\epsilon^{-1}(w^{\eta\epsilon})))^2}{\eta} \frac{\partial w^{\eta\epsilon}}{\partial x}\, dx\, dy + \int_{\Gamma_0} \theta_0 h w^{\eta\epsilon}\, dy.$$

For the first integral, two cases $G_\epsilon^{-1}(w^{\eta\epsilon}) \geqslant 0$ and $G_\epsilon^{-1}(w^{\eta\epsilon}) < 0$ are to be considered. Using Lemma 2.2, we deduce that $0 \leqslant B(G_\epsilon^{-1}(w^{\eta\epsilon})) \leqslant 1$ if $G_\epsilon^{-1}(w^{\eta\epsilon}) < 0$. Moreover, for $G_\epsilon^{-1}(w^{\eta\epsilon}) \geqslant 0$ we have $G_\epsilon^{-1} = \mathbb{I}d$ (the identity) so that

$$\int_\Omega h^3 \big(\nabla w^{\eta\epsilon}\big)^2 \, dx \\ \leqslant |\Omega|^{\frac{1}{2}} |\|h\|_{L^\infty(\Omega)} \left\| \frac{\partial w^{\eta\epsilon}}{\partial x} \right\|_{L^2(\Omega)} + \|h\|_{L^\infty(\Omega)} \big\|B\big(w^{\eta\epsilon}\big)\big\|_{L^2(\Omega)} \left\| \frac{\partial w^{\eta\epsilon}}{\partial x} \right\|_{L^2(\Omega)}$$

$$+\eta|\Omega|^{\frac{1}{2}}|\|h\|_{L^\infty(\Omega)}\left\|\frac{\partial w^{\eta\epsilon}}{\partial x}\right\|_{L^2(\Omega)}+\|h\|_{L^\infty(\Omega)}\|\theta_0\|_{L^2(\Gamma_0)}\left\|w^{\eta\epsilon}\right\|_{L^2(\Omega)};$$

therefore

$$\begin{aligned}&\min_{\{x\in[0,1]\}}\big(h^3(x)\big)\left\|w^{\eta\epsilon}\right\|_V^2\\&\quad\leqslant\|h\|_{L^\infty(\Omega)}\left(\frac{1}{\beta}\left\|w^{\eta\epsilon}\right\|_{L^2(\Omega)}+|\Omega|^{\frac{1}{2}}|(2+\eta)+\|\theta_0\|_{L^2(\Gamma_0)}\right)\left\|w^{\eta\epsilon}\right\|_V,\end{aligned}$$

i.e.,

$$\min_{\{x\in[0,1]\}}\big(h^3(x)\big)\left\|w^{\eta\epsilon}\right\|_V\leqslant\|h\|_{L^\infty(\Omega)}\left(\frac{1}{\beta}\left\|w^{\eta\epsilon}\right\|_{L^2(\Omega)}+|\Omega|^{\frac{1}{2}}|(2+\eta)+\|\theta_0\|_{L^2(\Gamma_0)}\right),$$

so

$$\left(\min_{\{x\in[0,1]\}}\big(h^3(x)\big)-\frac{1}{\beta}C_p\|h\|_{L^\infty(\Omega)}\right)\left\|w^{\eta\epsilon}\right\|_V\leqslant\|h\|_{L^\infty(\Omega)}\big(|\Omega|^{\frac{1}{2}}|(2+\eta)+\|\theta_0\|_{L^2(\Gamma_0)}\big).$$

It follows that if $\min_{\{x\in[0,1]\}}(h^3(x))-\frac{1}{\beta}C_p\|h\|_{L^\infty(\Omega)}>0$ we have

$$\left\|w^{\eta\epsilon}\right\|_V\leqslant C_1$$

with

$$C_1=\frac{\|h\|_{L^\infty(\Omega)}(3|\Omega|^{\frac{1}{2}}+\|\theta_0\|_{L^2(\Gamma_0)})}{\min_{\{x\in[0,1]\}}(h^3(x))-\frac{1}{\beta}C_p\|h\|_{L^\infty(\Omega)}}.\qquad\square$$

Theorem 2.5. *If* $\min_{\{x\in[0,1]\}}(h^3(x))-\frac{1}{\beta}C_p\|h\|_{L^\infty(\Omega)}>0$, *then there exists* $v^{\eta\epsilon}$ *in* V *such that*

$$\begin{aligned}&\int_\Omega h^3\nabla v^{\eta\epsilon}\nabla\phi\,dx\,dy-\int_\Omega hH_\eta\big(B\big(G_\epsilon^{-1}\big(v^{\eta\epsilon}\big)\big)\big)B\big(G_\epsilon^{-1}\big(v^{\eta\epsilon}\big)\big)\frac{\partial\phi}{\partial x}\,dx\,dy\\&\quad=\int_{\Gamma_0}\theta_0h\phi\,dy\quad\forall\phi\in V.\end{aligned}$$

The proof of this theorem will be based on a generalization of the Ky Fan's theorem which is briefly recalled in Lemma 2.10. This enables us to give a proof, adapted in our variational context, of Proposition 2.7 [9] which in turn leads to Theorem 2.5.

Notation 2.6. Let us denote by $\mathcal{F}(X)$ the family of all non-empty finite subsets of X and by $\mathcal{F}(X,x_0)$ all elements of $\mathcal{F}(X)$ containing x_0. Let us denote by $\mathrm{conv}(A)$ the convex hull of A, by $\overline{A}^X$ the closure of A in X and by $\mathrm{int}_X(A)$ the interior of A in X.

Proposition 2.7. [9] *Let E be a topological vector space and X be a non-empty convex subset of E; $\Psi,\Phi:X\times X\to\overline{\mathbb{R}}$ such that*:

(1) $\Psi(\chi,q)\leqslant\Phi(\chi,q)$ *for all* $\chi,q\in X$ *and* $\Phi(\chi,\chi)\leqslant 0$ *for all* $\chi\in X$.
(2) *For all* $A\in\mathcal{F}(X)$ *and all* $\chi\in\mathrm{conv}(A)$, $q\mapsto\Psi(\chi,q)$ *is lower semicontinuous on* $\mathrm{conv}(A)$.
(3) *For all* $q\in X$, *the set* $\{\chi\in X,\ \Phi(\chi,q)>0\}$ *is convex.*

(4) *For all* $A \in \mathcal{F}(X)$ *and all* $\chi, q \in \mathrm{conv}(A)$ *and for every net* $\{q_\alpha\}$ *converging in* X *to* q *with* $\Psi(t\chi + (1-t)q, q_\alpha) \leqslant 0$ *for all* α *and all* $t \in [0,1]$, *we have* $\Psi(\chi, q) \leqslant 0$.
(5) *There exists a non-empty closed and compact* K *of* X *and* $x_0 \in K$ *such that* $\Psi(x_0, q) > 0$ $\forall q \in X \backslash K$.

Then there exists $\bar{q} \in K$ *such that* $\Psi(\chi, \bar{q}) \leqslant 0$ $\forall \chi \in X$.

Remark 2.8. If the application $q \mapsto \Psi(\chi, q)$ is lower semicontinuous on X for all $\chi \in X$, then conditions (2) and (4) are fulfilled.

Definition 2.9. [9] $T : X \to 2^E$ is said to be a KKM-application if for all $A \in \mathcal{F}(X)$, $\mathrm{conv}(A) \subseteq \bigcup_{\chi \in A} T(\chi)$.

First, we recall the following lemma that is a generalization of Ky Fan's lemma.

Lemma 2.10. [11] *Let* X *be a non-empty convex subset included or equal to* E (*a topological vector space*) *and* $T : X \to 2^E$ *be a KKM-application, we suppose that there exists* x_0 *which belongs to* X *such that*:

(i) $\overline{T(x_0) \cap X}^X$ *is compact on* X.
(ii) $\forall A \in \mathcal{F}(X, x_0)$, $\forall \chi \in \mathrm{conv}(A)$, $T(\chi) \cap \mathrm{conv}(A)$ *is closed in* $\mathrm{conv}(A)$.
(iii) $\forall A \in \mathcal{F}(X, x_0)$, $\overline{X \cap (\bigcap_{\chi \in \mathrm{conv}(A)} T(\chi))}^X \cap \mathrm{conv}(A) = (\bigcap_{\chi \in \mathrm{conv}(A)} T(\chi)) \cap \mathrm{conv}(A)$.

Then $\bigcap_{\chi \in X} T(\chi) \neq \emptyset$.

Proof of Theorem 2.7. For all $\chi \in X$, let

$$T(\chi) = \{q \in X,\ \Psi(\chi, q) \leqslant 0\}.$$

Condition (5) implies that $T(x_0) \subseteq K$, i.e., $\overline{T(x_0)}^X$ is compact on X. Condition (2) implies that for any χ in $\mathrm{conv}(A)$, $T(\chi) \cap \mathrm{conv}(A)$ is closed in $\mathrm{conv}(A)$, then we will prove that T is a KKM-application. Let us suppose that T is not a KKM-application, then there exists A in $\mathcal{F}(X)$ and q_0 in $\mathrm{conv}(A)$ such that $q_0 \notin \bigcup_{\chi \in A} T(\chi)$ and so we get that $\Psi(\chi, q_0) > 0$ $\forall \chi \in A$. However, as $\{\chi \in X,\ \Psi(\chi, q_0) > 0\}$ is convex then $\mathrm{conv}(A) \subset \{\chi \in X,\ \Psi(\chi, q_0) > 0\}$ and $\Psi(q_0, q_0) > 0$ inducing $\Phi(q_0, q_0) > 0$ which is impossible. Hence T is a KKM-application. Remains to show that:

$$\overline{X \cap \Big(\bigcap_{\chi \in \mathrm{conv}(A)} T(\chi)\Big)}^X \cap \mathrm{conv}(A) = \Big(\bigcap_{\chi \in \mathrm{conv}(A)} T(\chi)\Big) \cap \mathrm{conv}(A), \quad \text{for all } A \text{ in } \mathcal{F}(X).$$

For any q in $\overline{X \cap (\bigcap_{\chi \in \mathrm{conv}(A)} T(\chi))}^X \cap \mathrm{conv}(A)$, there exists a sequence (q_α) such that q_α tends to q and q_α belongs to $X \cap (\bigcap_{\chi \in \mathrm{conv}(A)} T(\chi))$. However, as q_α belongs to $\bigcap_{\chi \in \mathrm{conv}(A)} T(\chi)$, then $\Psi(\chi, q_\alpha) \leqslant 0$ for all χ in $\mathrm{conv}(A)$, so that

$$\Psi\big(t\chi + (1-t)q, q_\alpha\big) \leqslant 0 \quad \text{for all } (\chi, q) \text{ in } \big(\mathrm{conv}(A)\big)^2,\ 0 \leqslant t \leqslant 1.$$

From (4) in Theorem 2.7 we get immediately

$$\Psi(\chi, q) \leqslant 0 \quad \forall \chi \in \mathrm{conv}(A),$$

so that

$$q \in \Bigl(\bigcap_{\chi \in \operatorname{conv}(A)} T(\chi)\Bigr) \cap \operatorname{conv}(A).$$

And by application of Lemma 2.10, there exists $\bar{q}$ in K such that $\bar{q}$ belongs to $T(\chi)$ for any χ in X, i.e., there exists $\bar{q} \in K$ such that $\Psi(\chi, \bar{q}) \leqslant 0\ \forall \chi \in X$. □

Proof of Proposition 2.5. Let us consider the space

$$E := \bigl(V, \|.\|_{H_0^1(\Omega)}\bigr)$$

endowed of its weak topology

$$X := \bigl\{u \in V,\ \|u\|_V \leqslant M\bigr\}$$

with $M > C_1$, in which C_1 is defined in Lemma 2.4 and the applications

$$\Psi(\chi, q) := \Phi(\chi, q) := \int_\Omega h^3 \nabla q \nabla (q-\chi)\, dx\, dy - \int_\Omega h F_{\eta\epsilon}(q) \frac{\partial}{\partial x}(q-\chi)\, dx\, dy - \int_{\Gamma_0} \theta_0 h (q-\chi)\, dy$$

with $F_{\eta\epsilon}(q) := H_\eta(B(G_\epsilon^{-1}(q)))B(G_\epsilon^{-1}(q))$ for all χ, q in V.

We will show that conditions of Theorem 2.7 are satisfied. The condition (1) is obviously satisfied and as the application $\chi \to \Psi(\chi, q)$ is linear then condition (3) is also fulfilled. For condition (5) it is sufficient to take

$$K := X = \bigl\{u \in V,\ \|u\|_V \leqslant M\bigr\}.$$

According to Remark 2.8, it is sufficient to demonstrate that the application $q \mapsto \Psi(\chi, q)$ is weakly lower semicontinuous in V to conclude that conditions (2) and (4) are satisfied. Indeed, let $q_n \rightharpoonup q$ in V, then $\int_{\Gamma_0} \theta_0 (q_n - \chi)\, dy \to \int_{\Gamma_0} \theta_0 (q-\chi)\, dy$ since $\theta_0 \in L^\infty(\Gamma_0)$. On the other hand, there exists a subsequence q_{n_k} such that $q_{n_k} \to q$ in $L^2(\Omega)$ and $\nabla q_{n_k} \rightharpoonup \nabla q$ in $L^2(\Omega)$. Therefore while using the Lebesgue's dominated convergence theorem, we find

$$\begin{aligned}\int_\Omega h F_{\eta\epsilon}(q_{n_k}) \frac{\partial (q_{n_k} - \chi)}{\partial x} &= \int_\Omega h F_{\eta\epsilon}(q_{n_k}) \frac{\partial q_{n_k}}{\partial x} - \int_\Omega h F_{\eta\epsilon}(q_{n_k}) \frac{\partial \chi}{\partial x} \\ &\to \int_\Omega h F_{\eta\epsilon}(q) \frac{\partial q}{\partial x} - \int_\Omega h F_{\eta\epsilon}(q) \frac{\partial \chi}{\partial x}.\end{aligned}$$

For the other term of $\Psi(\chi, q_{n_k})$ we have

$$\int_\Omega h^3 \nabla q_{n_k} \nabla (q_{n_k} - \chi) = \int_\Omega h^3 \nabla q_{n_k} \nabla q_{n_k} - \int_\Omega h^3 \nabla q_{n_k} \nabla \chi.$$

As ∇q_{n_k} weakly converges to ∇q in $L^2(\Omega)$, then $\int_\Omega h^3 \nabla q_{n_k} \nabla \chi$ converges to $\int_\Omega h^3 \nabla q \nabla \chi$. Moreover, as the application

$$z \mapsto \int_\Omega h^3 z^2$$

is a convex and weakly lower semicontinuous one in $L^2(\Omega)$ then

$$\underline{\lim}\left(\int_\Omega h^3\big((\nabla q_{n_k})^2-(\nabla q)^2\big)\right)\geqslant 0.$$

From Remark 2.8 and Theorem 2.7, there exists $v^{\eta\epsilon}$ in K such that for any χ in X $\Psi(\chi, v^{\eta\epsilon})\leqslant 0$. However, according to Lemma 2.4, $v^{\eta\epsilon}$ belongs to $\mathrm{int}_E(X)$ so that $\Psi(\chi, v^{\eta\epsilon})\leqslant 0$ for any χ. Choosing $\chi = v^{\eta\epsilon}+\gamma.\xi\in X$, for any ξ in V and convenient value of γ, we deduce that

$$\Psi\big(\xi, v^{\eta\epsilon}\big)=0 \quad \forall\xi\in V.$$

Then there exists a solution $v^{\eta\epsilon}$ of problem $(\mathcal{Q}_{\eta\epsilon})$ and in turn $u_{\eta\epsilon}=G_\epsilon^{-1}(v^{\eta\epsilon})$ is a solution of $(\mathcal{P}_{\eta\epsilon})$. □

2.3. Uniqueness of solution for the approached problems

Uniqueness of the solution of problem $(\mathcal{Q}_{\eta\epsilon})$ will be the direct consequence of the following lemma.

Lemma 2.11. *For any g_1 in V' and g_2 in $L^\infty(\Omega)$, there exists at most a solution of the problem*:

$$\begin{cases} \textit{Find } \phi \textit{ in } V \textit{ such that}\\ \displaystyle\int_\Omega h^3\nabla\phi\nabla\varphi=\int_\Omega hg_2\phi\frac{\partial\varphi}{\partial x}+\langle g_1,\varphi\rangle. \end{cases} \tag{2.7}$$

Proof. *Uniqueness*. Let ϕ_1 and ϕ_2 be two solutions for Eq. (2.7), κ be a (small) positive real parameter. We set

$$\pi_\kappa=\frac{\phi_3^+}{\phi_3^++\kappa},\quad \text{with } \phi_3=\phi_1-\phi_2,\ \phi_3^+=\max(0,\phi_3).$$

Making the difference of the two equations satisfied by ϕ_1 and ϕ_2, and multiplying the resulting equation by π_κ, we obtain

$$\int_\Omega \kappa h^3\left(\frac{\nabla\phi_3^+}{\phi_3^++\kappa}\right)^2 dx\,dy-\int_\Omega hg_2\frac{\partial\phi_3}{\partial x}\frac{\phi_3^+}{\phi_3^++\kappa}\,dx\,dy=0.$$

It follows that

$$\begin{aligned}\kappa\min_{x\in[0,1]}\big(h^3(x)\big)\left\|\frac{\nabla\phi_3^+}{\phi_3^++\kappa}\right\|^2_{L^2(\Omega)} &\leqslant \max_{x\in[0,1]}\big(h(x)\big)\|g_2\|_{L^\infty(\Omega)}\int_\Omega\left|\frac{\partial\phi_3}{\partial x}\frac{\phi_3^+}{\phi_3^++\kappa}\right|dx\,dy\\ &\leqslant \max_{x\in[0,1]}\big(h(x)\big)\|g_2\|_{L^\infty(\Omega)}\int_\Omega\left|\phi_3^+\frac{\partial}{\partial x}\left(\frac{\phi_3^+}{\phi_3^++\kappa}\right)\right|dx\,dy\\ &\leqslant \kappa\max_{x\in[0,1]}\big(h(x)\big)\|g_2\|_{L^\infty(\Omega)}\int_\Omega\left|\frac{\phi_3^+}{\phi_3^++\kappa}\left(\frac{\frac{\partial}{\partial x}\phi_3^+}{\phi_3^++\kappa}\right)\right|dx\,dy\\ &\leqslant \kappa\max_{x\in[0,1]}\big(h(x)\big)\|g_2\|_{L^\infty(\Omega)}\left\|\frac{\nabla\phi_3^+}{\phi_3^++\kappa}\right\|_{L^2(\Omega)},\end{aligned}$$

i.e.,

$$\int_\Omega \left(\log\left(1+\frac{\phi_3^+}{\kappa}\right)\right)^2 dx\,dy \leqslant C$$

where C is a constant that does not depend on κ. Then letting κ tend to zero, we will necessarily have $\phi_3 \leqslant 0$ a.e. in Ω. While substituting ϕ_3 for $-\phi_3$ we get $\phi_3 \geqslant 0$ a.e. in Ω. The uniqueness is proved.

Existence. Let the application

$$L_\sigma : V \to V',$$
$$\phi \mapsto L_\sigma(\phi) = \sigma\phi - \operatorname{div}\big(h^3\nabla\phi\big) - hg_2\frac{\partial\phi}{\partial x}.$$

We choose σ such that $\langle L_\sigma(\phi), \phi\rangle_{V',V} \geqslant \alpha_1\|\phi\|_V^2$ (with $\alpha_1 > 0$). However, ϕ is a solution of (2.7) if and only if

$$\phi - \sigma L_\sigma^{-1} \circ I\phi = L_\sigma^{-1} g_1,$$

in which the operator $I : V \to V'$ is defined by

$$\langle I\phi_1, \phi_2\rangle_{V',V} = \int_\Omega \phi_1\phi_2\,dx\,dy \quad \forall\phi_1, \phi_2 \in V.$$

As V is compactly embedded in $L^2(\Omega)$, then I is completely continuous. On the other hand, σL_σ^{-1} is continuous, then $\sigma L_\sigma^{-1} \circ I$ is completely continuous. And by applying the Fredholm's alternative [6] the result follows. □

Theorem 2.12. *The solutions of both problems* $(\mathcal{Q}_{\eta\epsilon})$ *and* $(\mathcal{P}_{\eta\epsilon})$ *are unique.*

Proof. Let $(v^{\eta\epsilon})^1$ and $(v^{\eta\epsilon})^2$ be two solutions of $(\mathcal{P}_{\eta\epsilon})$; then we have

$$\int_\Omega h^3\nabla\big(\big(v^{\eta\epsilon}\big)^1 - \big(v^{\eta\epsilon}\big)^2\big)\nabla\phi - \int_\Omega h\big(F_{\eta\epsilon}\big(\big(v^{\eta\epsilon}\big)^1\big) - F_{\eta\epsilon}\big(\big(v^{\eta\epsilon}\big)^2\big)\big)\frac{\partial\phi}{\partial x} = 0 \quad \forall\phi \in V, \tag{2.8}$$

with $F_{\eta\epsilon}(.) = H_\eta(B(G_\epsilon^{-1}(.)))B(G_\epsilon^{-1}(.))$.

$F_{\eta\epsilon}$ is a Lipschitz function; indeed

$$F_{\eta\epsilon}(v) = \begin{cases} B(v) & \text{if } \{v \geqslant 0\},\\ B(G_\epsilon^{-1}(v)) & \text{if } \{v < 0\} \cap \{\eta < B(G_\epsilon^{-1}(v)) < 1\},\\ \frac{(B(G_\epsilon^{-1}(v)))^2}{\eta} & \text{if } \{v < 0\} \cap \{0 < B(G_\epsilon^{-1}(v)) \leqslant \eta\},\\ 0 & \text{if } \{v < 0\} \cap \{B(G_\epsilon^{-1}(v)) \leqslant 0\}, \end{cases}$$

which is continuous, differentiable by intervals, its derivative is bounded in each interval, so that $F_{\eta\epsilon}$ is a Lipschitz function.

Let us consider now

$$\widetilde{F} = \begin{cases} \frac{F_{\eta\epsilon}((v^{\eta\epsilon})^1) - F_{\eta\epsilon}((v^{\eta\epsilon})^2)}{(v^{\eta\epsilon})^1 - (v^{\eta\epsilon})^2} & \text{if } (v^{\eta\epsilon})^1 \neq (v^{\eta\epsilon})^2,\\ 0 & \text{if } (v^{\eta\epsilon})^1 = (v^{\eta\epsilon})^2, \end{cases}$$

which belongs to $L^\infty(\Omega)$. Let $w=(v^{\eta\epsilon})^1-(v^{\eta\epsilon})^2$, then Eq. (2.8) can be rewritten as

$$\int_\Omega h^3\nabla w\nabla\phi-\int_\Omega h\widetilde{F}w\frac{\partial\phi}{\partial x}=0\quad\forall\phi\in V.$$

As $w=0$ is an obvious solution, according to Lemma 2.11, we deduce that $(v^{\eta\epsilon})^1=(v^{\eta\epsilon})^2$ and the uniqueness is proved. □

Proposition 2.13. *The solution of the problem* $(\mathcal{P}_\epsilon)$ *defined by* (2.2) *has at most a solution.*

Proof. Proof is the same as that of the previous proposition, pointing out that G_ϵ^{-1} is a Lipschitz function. □

2.4. Existence of a solution for $(\mathcal{P})$

Theorem 2.14. *If* $\min_{\{x\in[0,1]\}}(h^3(x))-\frac{1}{\beta}C_p\|h\|_{L^\infty(\Omega)}>0$*, then problem* $(\mathcal{P})$ *admits at least one solution.*

Proof. First, let us fix ϵ and let η tends to zero. As $v^{\eta\epsilon}$ is bounded in $H^1(\Omega)$ independently of η and ϵ (according to Lemma 2.4), there exists v^ϵ in $H^1(\Omega)$ such that $v^{\eta\epsilon}$ weakly converges to v^ϵ in $H^1(\Omega)$ as η tends to zero (a subsequence), and $v^{\eta\epsilon}$ strongly converges to v^ϵ in $L^2(\Omega)$. Then $G_\epsilon^{-1}(v^{\eta\epsilon})$ strongly converges to $G_\epsilon^{-1}(v^\epsilon)$ in $L^2(\Omega)$. Moreover, $H_\eta(B(G_\epsilon^{-1}(v^{\eta\epsilon})))$ weakly converges to δ^ϵ in $L^2(\Omega)$, with $0\leqslant\delta^\epsilon\leqslant 1$.

Passing to the limit ($\eta\to 0$) in Eq. (2.4), we get

$$\int_\Omega h^3\nabla v^\epsilon\nabla\phi\,dx\,dy-\int_\Omega h\delta^\epsilon B\big(G_\epsilon^{-1}\big(v^\epsilon\big)\big)\frac{\partial\phi}{\partial x}\,dx\,dy=\int_{\Gamma_0}\theta_0 h\phi\,dy\quad\forall\phi\in V.\tag{2.9}$$

It is easy to prove that:

$$\delta^\epsilon=0\quad\text{on }\big\{B\big(G_\epsilon^{-1}\big(v^\epsilon\big)\big)<0\big\},$$
$$\delta^\epsilon=1\quad\text{on }\big\{B\big(G_\epsilon^{-1}\big(v^\epsilon\big)\big)>0\big\},$$

so that $\delta^\epsilon B(G_\epsilon^{-1}(v^\epsilon))=[B(G_\epsilon^{-1}(v^\epsilon))]^+$ and (2.9) can be rewritten as

$$\int_\Omega h^3\nabla v^\epsilon\nabla\phi\,dx\,dy-\int_\Omega h\big[B\big(G_\epsilon^{-1}\big(v^\epsilon\big)\big)\big]^+\frac{\partial\phi}{\partial x}\,dx\,dy=\int_{\Gamma_0}\theta_0 h\phi\,dy\quad\forall\phi\in V.\tag{2.10}$$

Defining $u^\epsilon=G_\epsilon^{-1}(v^\epsilon)$, we get that u^ϵ is a solution of the problem $(\mathcal{P}_\epsilon)$.

Let us now pass to the limit on ϵ in Eq. (2.10). Due to the definition of G_ϵ, we have

$$G_\epsilon^{-1}\big(v^\epsilon\big)=G_\epsilon^{-1}\big(\big(v^\epsilon\big)^+\big)+G_\epsilon^{-1}\big(-\big(v^\epsilon\big)^-\big)=\big(v^\epsilon\big)^++G_\epsilon^{-1}\big(-\big(v^\epsilon\big)^-\big).$$

Moreover, according to Lemma 2.2 and the property of B issued from (1.3):

$$\begin{aligned}\big(B(u_\epsilon)\big)^+&=B(u_\epsilon)=B\big(G_\epsilon^{-1}\big(v^\epsilon\big)\big)=B\big(G_\epsilon^{-1}\big(-\big(v^\epsilon\big)^-\big)+\big(v^\epsilon\big)^+\big)\\&=B\big(G_\epsilon^{-1}\big(-\big(v^\epsilon\big)^-\big)+B\big(\big(v^\epsilon\big)^+\big)\big)-1,\end{aligned}$$

so that (2.10) is rewritten:

$$\int_\Omega h^3\nabla v^\epsilon\nabla\phi\,dx\,dy-\int_\Omega hB\big(G_\epsilon^{-1}\big(-\big(v^\epsilon\big)^-\big)+\big(v^\epsilon\big)^+\big)\frac{\partial\phi}{\partial x}\,dx\,dy$$
$$=\int_{\Gamma_0}\theta_0 h\phi\,dy\quad\forall\phi\in V.\tag{2.11}$$

As $v^{\eta\epsilon}$ is bounded in V, it is also true for v^ϵ so that there exists v in V, weak limit of v^ϵ. Moreover $(v^\epsilon)^+$ strongly converges towards v^+ and so $B((v^\epsilon)^+)$ strongly converges towards $B(v^+)$ in $L^2(\Omega)$. As $B(u^\epsilon)\geqslant 0$ and $B(0)=1$, then $0\leqslant B(G_\epsilon^{-1}(-(v^\epsilon)^-))\leqslant 1$, so that there exists γ, with $0\leqslant\gamma\leqslant 1$, weak limit in $L^2(\Omega)$ of $B(G_\epsilon^{-1}(-(v^\epsilon)^-))$. While passing to the limit ($\epsilon\to 0$), we get

$$\int_\Omega h^3\nabla v\nabla\phi\,dx\,dy-\int_\Omega h\big(B\big(v^+\big)+\gamma-1\big)\frac{\partial\phi}{\partial x}\,dx\,dy=\int_{\Gamma_0}\theta_0 h\phi\,dy\quad\forall\phi\in V.\tag{2.12}$$

However, $v^\epsilon=G_\epsilon(u^\epsilon)\geqslant G_\epsilon(-1)$ which tends to zero, so $v=v^+$.

Defining $u=v+B^{-1}(\gamma)$, i.e., $u^+=v$ and $B(u)=B(v^+)+\gamma-1$, then (2.12) can be written as

$$\int_\Omega h^3\nabla u^+\nabla\phi\,dx\,dy-\int_\Omega hB(u)\frac{\partial\phi}{\partial x}\,dx\,dy=\int_{\Gamma_0}\theta_0 h\phi\,dy\quad\forall\phi\in V,\tag{2.13}$$

which is exactly the equation defining $(\mathcal{P})$ in Section 2, as the boundary condition satisfied by v^ϵ implies that u belongs to V. □

Remark 2.15. Using Theorem 2.7 induces an existence condition (2.5) somewhat more general than the one found in [8] and in which the proof is based on a fixed point technique.

3. Uniqueness of solution

3.1. A uniqueness result for the problem $(\mathcal{P})$

To get the result of uniqueness for the problem $(\mathcal{P})$, we will use a technique already developed by [7] for elliptic problems with non-linear convection terms and [3,17] for various lubrication problems. Although the general lines of the proof are identical to the ones of the preceding references, the proof has some distinct features; test functions have to be modified and the study of the limit of the right-hand side of the variation with respect to the parameters λ, μ needs specific calculation due to function $B(u)$.

The proof will be made in several steps, but before this, we will introduce the notation

$$k=B(u).\tag{3.1}$$

Then the problem $(\mathcal{P})$ can be written under the following form:

$$\begin{cases}\text{Let } \theta_0\in L^\infty(\Gamma_0),\ 0\leqslant\theta_0\leqslant 1;\ \text{find } (k,u^+)\in L^2(\Omega)\times V \text{ verifying } k\geqslant 0 \text{ and}\\ \displaystyle\int_\Omega h^3\nabla u^+\nabla\phi\,dx\,dy-\int_\Omega hk\frac{\partial\phi}{\partial x}\,dx\,dy=\int_{\Gamma_0}\theta_0 h\phi\,dy\quad\forall\phi\in V.\end{cases}\tag{3.2}$$

Let (k_1,u_1^+) and (k_2,u_2^+) be two solutions of the problem $(\mathcal{P})$, respectively associated to data θ_0^1 and θ_0^2.

First step. Let us consider that solutions (k_1, u_1^+) and (k_2, u_2^+) depend on different variables under the following form:

$$u_1^+ = u_1^+(x_1, y_1), \quad k_1 = k_1(x_1, y_1), \quad X_1 = (x_1, y_1), \quad \Omega_1 = \Omega(X_1), \quad \Gamma_0^1 = \Gamma_0(X_1),$$
$$u_2^+ = u_2^+(x_2, y_2), \quad k_2 = k_2(x_2, y_2), \quad X_2 = (x_2, y_2), \quad \Omega_2 = \Omega(X_2), \quad \Gamma_0^2 = \Gamma_0(X_2),$$

and we consider the set $\Upsilon = \Omega_1 \times \Omega_2$.

We define the auxiliary functions:

$$\begin{aligned}
&\varkappa = \varkappa(r) \in C_c^\infty(0, 1), \quad \varkappa \geqslant 0,\\
&\varsigma = \varsigma(r) \in C_c^\infty(\mathbb{R}), \quad \varsigma \geqslant 0 \quad \text{and} \quad \operatorname{supp}(\varsigma) \subset [-1, 1],\\
&\widetilde{\varsigma} = \widetilde{\varsigma}(r) \in C_c^\infty(\mathbb{R}), \quad \widetilde{\varsigma} \geqslant 0 \quad \text{and} \quad \operatorname{supp}(\widetilde{\varsigma}) \subset [-1, 1],\\
&\varsigma_\lambda = \varsigma_\lambda(r) = \frac{1}{\lambda}\varsigma\left(\frac{r}{\lambda}\right) \in C_c^\infty(\mathbb{R}),\\
&\widetilde{\varsigma}_\lambda = \widetilde{\varsigma}_\lambda(r) = \frac{1}{\lambda}\widetilde{\varsigma}\left(\frac{r}{\lambda}\right) \in C_c^\infty(\mathbb{R}),
\end{aligned}$$

with λ being small enough ($0 < \lambda < \operatorname{dist}(\operatorname{supp}(\varkappa), \partial[0, 1])$, where dist denotes the usual distance between sets of real numbers).

We construct also the test function

$$\xi_\mu(X_1, X_2) = \min\left[\frac{(u_1^+(X_1) - u_2^+(X_2))^+}{\mu}, \tau(X_1, X_2)\right]$$

with $\mu \in \mathbb{R}_+^*$ and

$$\tau(X_1, X_2) = \varkappa\left(\frac{y_1 + y_2}{2}\right)\varsigma_\lambda\left(\frac{y_1 - y_2}{2}\right)\widetilde{\varsigma}_\lambda\left(\frac{x_1 - x_2}{2}\right).$$

Proposition 3.1. *The function $\xi_\mu(., X_2)$ belongs to V for all X_2 in Ω_2, and the function $\xi_\mu(X_1, .)$ belongs to V for all X_1 in Ω_1.*

Proof. It is obvious that $\xi_\mu(., X_2) \in H^1(\Omega_1)$ for all $X_2 \in \Omega_2$ and $\xi_\mu(X_1, .) \in H^1(\Omega_2)$ for all $X_1 \in \Omega_1$, and that $\xi_\mu(., X_2)$ is identically zero on Γ.

To finish the proof it is sufficient to prove that $\xi_\mu(X_1, .)$ is identically zero on Γ; indeed, if (x_2, y_2) belongs to Γ, there are two possibilities:

- $|y_1 - y_2| \geqslant 2\lambda \Rightarrow \varsigma_\lambda(\frac{y_1 - y_2}{2}) = 0 \Rightarrow \xi_\mu(X_1, X_2) = 0$;
- $|y_1 - y_2| < 2\lambda$. In this case, we have:

$$\operatorname{dist}\left(\frac{y_1 + y_2}{2}, \partial[0, 1]\right) < \lambda \leqslant \operatorname{dist}\bigl(\operatorname{supp}(\varkappa), \partial[0, 1]\bigr).$$

We deduce that $\frac{y_1 + y_2}{2} \notin \operatorname{supp}(\varkappa)$ and $\xi_\mu(X_1, X_2) = 0$. $\square$

Second step. We introduce the notations

$$\nabla_1 = \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial y_1}\right), \qquad \nabla_2 = \left(\frac{\partial}{\partial x_2}, \frac{\partial}{\partial y_2}\right).$$

Then (k_1, u_1^+) verify the equation

$$\int_{\Omega_1} h^3(x_1)\nabla_1 u_1^+(X_1)\nabla_1\xi_\mu\,dX_1 - \int_{\Omega_1} h(x_1)k_1(X_1)\frac{\partial\xi_\mu}{\partial x_1}\,dX_1 = \int_{\Gamma_0^1} \theta_0^1(y_1)h(0).\xi_\mu\,dy_1.$$

The integration by the variable X_2 on Ω_2 leads to

$$\int_{\Upsilon} h^3(x_1)\nabla_1 u_1^+(X_1)\nabla_1\xi_\mu\,dX_1\,dX_2 - \int_{\Upsilon} h(0)k_1(X_1)\frac{\partial\xi_\mu}{\partial x_1}\,dX_1\,dX_2$$
$$= \int_{\Gamma_0^1\times\Omega_2} \theta_0^1(y_1)h(x_1)\xi_\mu\,dy_1\,dX_2. \tag{3.3}$$

With analogous computations (k_2, u_2^+) satisfy the equation

$$\int_{\Upsilon} h^3(x_2)\nabla_2 u_2^+(X_2)\nabla_2\xi_\mu\,dX_2\,dX_1 - \int_{\Upsilon} h(x_2)k_2(X_2)\frac{\partial\xi_\mu}{\partial x_2}\,dX_2\,dX_1$$
$$= \int_{\Gamma_0^2\times\Omega_1} \theta_0^2(y_2)h(0)\xi_\mu\,dy_2\,dX_1. \tag{3.4}$$

However, from the boundary conditions ξ_μ, we have:

$$\int_{\Upsilon} h^3(x_1)\nabla_1 u_1^+(X_1)\nabla_2\xi_\mu\,dX_2\,dX_1 = 0,$$
$$\int_{\Upsilon} h^3(x_2)\nabla_2 u_1^+(X_2)\nabla_1\xi_\mu\,dX_2\,dX_1 = 0,$$
$$\int_{\Upsilon} h(x_1)k_1(X_1)\frac{\partial\xi_\mu}{\partial x_2}\,dX_1\,dX_2 = 0,$$
$$\int_{\Upsilon} h(x_2)k_2(X_2)\frac{\partial\xi_\mu}{\partial x_1}\,dX_2\,dX_1 = 0,$$

while using the above relations and relations (3.3) and (3.4), we find the following integral identity

$$\int_{\Upsilon} \big[h^3(x_1)(\nabla_1+\nabla_2)u_1^+(X_1) - h^3(x_2)(\nabla_1+\nabla_2)u_1^+(X_2)\big](\nabla_1+\nabla_2)\xi_\mu\,dX_2\,dX_1$$
$$= \int_{\Upsilon} \big[h(x_1)k_1(X_1) - h(x_2)k_1(X_2)\big]\left(\frac{\partial\xi_\mu}{\partial x_1} + \frac{\partial\xi_\mu}{\partial x_2}\right)dX_2\,dX_1$$
$$+ \int_{\Gamma_0^1\times\Omega_2} \theta_0^1(y_1)h(0)\xi_\mu\,dy_1\,dX_2 - \int_{\Gamma_0^2\times\Omega_1} \theta_0^2(y_2)h(0)\xi_\mu\,dy_2\,dX_1 \tag{3.5}$$

which finishes the second step.

Third step. We introduce the following change of variables $t=(t_1,t_2)$, $z=(z_1,z_2)$:

$$z=\frac{X_1+X_2}{2},\qquad t=\frac{X_1-X_2}{2} \tag{3.6}$$

for integrals on Υ;

$$\begin{aligned} &z_2=\frac{y_1+y_2}{2},\qquad t_2=\frac{y_1-y_2}{2},\\ &z_1=x_2\quad \text{for the integral on } \Gamma_0^1\times\Omega_2,\\ &z_1=x_1\quad \text{for the integral on } \Gamma_0^2\times\Omega_1, \end{aligned} \tag{3.7}$$

and we denote by Λ the same image of the sets $\Gamma_0^1\times\Omega_2$ and $\Gamma_0^2\times\Omega_1$. In the new variables we have $u_1^+(X_1)=u_1^+(z+t)$, $k_1(X_1)=k_1(z+t)$, $u_2^+(X_2)=u_2^+(z-t)$ and $k_2(X_2)=k_2(z-t)$. Then the relation (3.5) can be written (for new variables) as follows:

$$\begin{aligned} &\int_\Upsilon \big(h^3(z_1+t_1)\nabla_z u_1^+(z+t)-h^3(z_1-t_1)\nabla_z u_2^+(z-t)\big)\nabla_z\xi_\mu(z,t)\,dt\,dz\\ &\quad=\int_\Upsilon \big(h(z_1+t_1)k_1(z+t)-h(z_1-t_1)k_2(z-t)\big)\frac{\partial\xi_\mu(z,t)}{\partial z_1}\,dt\,dz\\ &\qquad+\int_\Lambda \theta_0^1(z_2+t_2)h(0)\xi_\mu\,dz_2\,dt_2\,dz_1-\int_\Lambda \theta_0^2(z_2-t_2)h(0)\xi_\mu\,dz_2\,dt_2\,dz_1 \end{aligned} \tag{3.8}$$

in which we have omitted the constant due to the coordinates transform.

Fourth step. For sake of clearness, we enumerate the integrals and we transform the expression (3.8) in

$$I_1=I_2+I_3. \tag{3.9}$$

We will study each of the integrals I_1, I_2, I_3 and their behaviour when limits are taken with respect of parameters $\lambda,\ \mu$.

Let us begin with the integral I_3:

$$\begin{aligned} I_3&=\left(\int_\Lambda \theta_0^1(z_2+t_2)h(0)\xi_\mu\,dz_2\,dt_2\,dz_1-\int_\Lambda \theta_0^2(z_2+t_2)h(0)\xi_\mu\,dz_2\,dt_2\,dz_1\right)\\ &\quad+\left(\int_\Lambda \theta_0^2(z_2+t_2)h(0)\xi_\mu\,dz_2\,dt_2\,dz_1-\int_\Lambda \theta_0^2(z_2-t_2)h(0)\xi_\mu\,dz_2\,dt_2\,dz_1\right)\\ &=I_{3,1}+I_{3,2}. \end{aligned}$$

We remark that if $\theta_0^1\leqslant\theta_0^2$ then

$$I_{3,1}\leqslant 0. \tag{3.10}$$

Now, we suppose that θ_0^2 is lipschitzian, $\operatorname{supp}(\varsigma_\lambda)\subset[-\lambda,\lambda]$ and $(\widetilde{\varsigma}_\lambda)$ can be chosen as an even function, there exists a constant c such that

$$\Big|\lim_{\mu\to 0} I_{3,2}\Big|\leqslant c\lambda\int_\Lambda \varkappa(z_2)\varsigma_\lambda(t_2)\widetilde{\varsigma}_\lambda(t_1)\,dt\,dz_2.$$

The integral being bounded, we obtain

$$\lim_{\lambda\to 0}\Big(\lim_{\mu\to 0} I_{3,2}\Big)=0. \tag{3.11}$$

Let us consider now the integral I_2: While coming back to the change of variables (3.1) and using the fact that $B(u)=B(u^+)+B(u^-)-1$, we have

$$\begin{aligned}
I_2 &= \int_{\Upsilon}\big(h(z_1+t_1)k_1(z+t)-h(z_1-t_1)k_2(z-t)\big)\frac{\partial\xi_\mu(z,t)}{\partial z_1}\,dt\,dz\\
&= \int_{\Upsilon}\big(h(z_1+t_1)k_1(z+t)-h(z_1-t_1)k_2(z-t)\big)\frac{\partial}{\partial z_1}\min\bigg(\frac{(u_1^+-u_2^+)^+}{\mu},\varkappa\varsigma_\lambda\widetilde{\varsigma}_\lambda\bigg)\,dt\,dz\\
&= \int_{\Upsilon}\big(h(z_1+t_1)B\big(u_1(z+t)\big)-h(z_1-t_1)B\big(u_2(z-t)\big)\big)\\
&\qquad\times\frac{\partial}{\partial z_1}\min\bigg(\frac{(u_1^+-u_2^+)^+}{\mu},\varkappa\varsigma_\lambda\widetilde{\varsigma}_\lambda\bigg)\,dt\,dz\\
&= \int_{\Upsilon}\big(h(z_1+t_1)B\big(u_1^+\big)-h(z_1-t_1)B\big(u_2^+\big)\big)\frac{\partial}{\partial z_1}\min\bigg(\frac{(u_1^+-u_2^+)^+}{\mu},\varkappa\varsigma_\lambda\widetilde{\varsigma}_\lambda\bigg)\,dt\,dz\\
&\quad+\int_{\Upsilon}\big(h(z_1+t_1)\big(B\big(u_1^-\big)-1\big)-h(z_1-t_1)\big(B\big(u_2^-\big)-1\big)\big)\\
&\qquad\times\frac{\partial}{\partial z_1}\min\bigg(\frac{(u_1^+-u_2^+)^+}{\mu},\varkappa\varsigma_\lambda\widetilde{\varsigma}_\lambda\bigg)\,dt\,dz\\
&= I_{2,1}-I_{2,2}.
\end{aligned}$$

Applying Green's formula we obtain:

$$I_{2,1}=-\int_{\Upsilon}\frac{\partial}{\partial z_1}\big(h(z_1+t_1)B\big(u_1^+\big)-h(z_1-t_1)B\big(u_2^+\big)\big)\min\bigg(\frac{(u_1^+-u_2^+)^+}{\mu},\varkappa\varsigma_\lambda\widetilde{\varsigma}_\lambda\bigg)\,dt\,dz,$$

and passing to the limit on μ:

$$\begin{aligned}
\lim_{\mu\to 0} I_{2,1} &= -\int_{\Upsilon}\frac{\partial}{\partial z_1}\big(h(z_1+t_1)B\big(u_1^+\big)-h(z_1-t_1)B\big(u_2^+\big)\big)\varkappa(z_2)\varsigma_\lambda(t_2)\widetilde{\varsigma}_\lambda(t_1)\,dt\,dz\\
&= \int_{\Upsilon}\big(h(z_1+t_1)B\big(u_1^+\big)-h(z_1-t_1)B\big(u_2^+\big)\big)\frac{\partial}{\partial z_1}\big(\varkappa(z_2)\varsigma_\lambda(t_2)\widetilde{\varsigma}_\lambda(t_1)\big)\,dt\,dz\\
&= 0
\end{aligned}$$

where we have used Green's formula again.

Let us consider the two sets

$$A_\mu=\big[\big(u_1^+-u_2^+\big)^+>\mu\varkappa\varsigma_\lambda\widetilde{\varsigma}_\lambda\big],\qquad D_\mu=\big[0<\big(u_1^+-u_2^+\big)\leqslant\mu\varkappa\varsigma_\lambda\widetilde{\varsigma}_\lambda\big].$$

Then we can write

$$I_{2,2} = \int_{\Upsilon} \big(h(z_1+t_1)\big(B\big(u_1^-\big)-1\big) - h(z_1-t_1)\big(B\big(u_2^-\big)-1\big)\big) \times \frac{\partial}{\partial z_1} \min\left(\frac{(u_1^+ - u_2^+)^+}{\mu}, \varkappa\varsigma_\lambda\widetilde{\varsigma}_\lambda\right) dt\,dz$$
$$= \int_{D_\mu} \big(h(z_1+t_1)\big(B\big(u_1^-\big)-1\big) - h(z_1-t_1)\big(B\big(u_2^-\big)-1\big)\big) \frac{\partial}{\partial z_1} \frac{(u_1^+ - u_2^+)^+}{\mu}\, dt\,dz.$$

As $(u_1^+ - u_2^+)^+ = 0$ as soon as $u_1 \leqslant 0$ and $(h(z_1+t_1)(B(u_1^-)-1) - h(z_1-t_1)(B(u_2^-)-1)) = 0$ if $u_2 > 0$, we get

$$I_{2,2} = -\int_{D_\mu} h(z_1-t_1)\big(B\big(u_2^-\big)-1\big) \frac{\partial}{\partial z_1}\left(\frac{u_1^+}{\mu}\right) dt\,dz.$$

Coming back to the initial variables, we obtain

$$I_{2,2} = -\int_{D_\mu} h(x_2)\big(B\big(u_2^-\big)-1\big)(X_2) \frac{\partial}{\partial x_1}\left(\frac{u_1^+}{\mu}\right) dX_1\,dX_2$$
$$= -\int_{\Upsilon} h(x_2)\big(B\big(u_2^-\big)-1\big)(X_2) \frac{\partial}{\partial x_1} \min\left(\frac{u_1^+}{\mu}, \varkappa\varsigma_\lambda\widetilde{\varsigma}_\lambda\right) dX_1\,dX_2$$
$$- \int_{A_\mu} h(x_2)\big(B\big(u_2^-\big)-1\big)(X_2) \frac{\partial \varkappa\varsigma_\lambda\widetilde{\varsigma}_\lambda}{\partial x_1}\, dX_1\,dX_2$$
$$= \int_{D_\mu} h(x_2)\big(B\big(u_2^-\big)-1\big)(X_2) \frac{\partial \varkappa\varsigma_\lambda\widetilde{\varsigma}_\lambda}{\partial x_1}\, dX_1\,dX_2.$$

As the function $h(x_2)(B(u_2^-)-1)(X_2)\frac{\partial \varkappa\varsigma_\lambda\widetilde{\varsigma}_\lambda}{\partial x_1}$ is bounded for each λ, we conclude

$$\lim_{\mu\to 0} |I_{2,2}| \leqslant \lim_{\mu\to 0} C_1|D_\mu| = 0$$

then

$$\lim_{\mu\to 0} I_2 = 0. \tag{3.12}$$

At last we can study the integral I_1: By introducing sets A_μ and D_μ as in the previous paragraph, we can write:

$$I_1 = \int_{A_\mu} \big(h^3(z_1+t_1)\nabla_z u_1^+(z+t) - h^3(z_1-t_1)\nabla_z u_2^+(z-t)\big)\nabla_z(\varkappa\varsigma_\lambda\widetilde{\varsigma}_\lambda)\, dt\,dz$$
$$+ \int_{D_\mu} \big(h^3(z_1+t_1)\nabla_z u_1^+(z+t) - h^3(z_1-t_1)\nabla_z u_2^+(z-t)\big)\nabla_z\left(\frac{u_1^+ - u_2^+}{\mu}\right) dt\,dz$$
$$= I_{1,1} + I_{1,2}. \tag{3.13}$$

Going back to the initial variables in $I_{1,2}$, we have

$$I_{1,2} = \int\limits_{D_\mu} \left(h^3(x_1) \left| \nabla_1 \frac{u_1^+}{\mu} \right|^2 + h^3(x_2) \left| \nabla_2 \frac{u_2^+}{\mu} \right|^2 \right)$$
$$- \int\limits_{D_\mu} h^3(x_1) \nabla_1 u_1^+ \nabla_2 \left(\frac{u_2^+}{\mu} \right) - \int\limits_{D_\mu} h^3(x_2) \nabla_2 u_2^+ \nabla_1 \left(\frac{u_1^+}{\mu} \right).$$

The first integral in the right-hand side is a positive one. Since $h^3(x_1)\nabla_1 u_1^+$ does not rely on X_2 and taking the boundary conditions for $\min((u_{1-}^+ u_2^+)^+/\mu, \varkappa\varsigma_\lambda\widetilde{\varsigma}_\lambda)$ into account, the second integral is rewritten as

$$- \int\limits_{D_\mu} h^3(x_1) \nabla_1 u_1^+ \nabla_2 \left(\frac{u_2^+}{\mu} \right)$$
$$= \int\limits_{D_\mu} h^3(x_1) \nabla_1 u_1^+ \nabla_2 \left(\frac{u_{1-}^+ u_2^+}{\mu} \right)$$
$$= \int\limits_{\Upsilon} h^3(x_1) \nabla_1 u_1^+ \nabla_2 \min\left(\frac{(u_{1-}^+ u_2^+)^+}{\mu}, \varkappa\varsigma_\lambda\widetilde{\varsigma}_\lambda \right) - \int\limits_{A_\mu} h^3(x_1) \nabla_1 u_1^+ \nabla_2 (\varkappa\varsigma_\lambda\widetilde{\varsigma}_\lambda)$$
$$= \int\limits_{\Upsilon} h^3(x_1) \nabla_1 u_1^+ \nabla_2 \min\left(\frac{(u_{1-}^+ u_2^+)^+}{\mu}, \varkappa\varsigma_\lambda\widetilde{\varsigma}_\lambda \right) - \int\limits_{\Upsilon} h^3(x_1) \nabla_1 u_1^+ \nabla_2 (\varkappa\varsigma_\lambda\widetilde{\varsigma}_\lambda)$$
$$+ \int\limits_{D_\mu} h^3(x_1) \nabla_1 u_1^+ \nabla_2 (\varkappa\varsigma_\lambda\widetilde{\varsigma}_\lambda)$$
$$= \int\limits_{D_\mu} h^3(x_1) \nabla_1 u_1^+ \nabla_2 (\varkappa\varsigma_\lambda\widetilde{\varsigma}_\lambda).$$

Now, by Hölder's inequality and since $\lim_{\mu\to 0} |D_\mu| = 0$ we conclude

$$\lim_{\mu\to 0} \left| \int\limits_{D_\mu} h^3(x_1) \nabla_1 u_1^+ \nabla_2 \left(\frac{u_2^+}{\mu} \right) \right| = 0.$$

In a similar way, the third integral in $I_{1,2}$ is such that

$$\lim_{\mu\to 0} \left| \int\limits_{D_\mu} h^3(x_2) \nabla_2 u_2^+ \nabla_1 \left(\frac{u_1^+}{\mu} \right) \right| = 0,$$

and then

$$\lim_{\mu\to 0} I_{1,2} \geqslant 0. \tag{3.14}$$

At last taking into account relations (3.9)–(3.14),

$$\lim_{\lambda\to 0} \Big(\lim_{\mu\to 0} I_{1,1} \Big) \leqslant 0. \tag{3.15}$$

Let us study more precisely the behaviour of $I_{1,1}$ from the definition (3.13): By Lebesgue theorem, we get

$$\lim_{\mu\to 0} I_{1,1} = \int_{\Upsilon} \left(h^3(z_1+t_1) \frac{\partial u_1^+(z+t)}{\partial z_2} - h^3(z_1-t_1) \frac{\partial u_2^+(z-t)}{\partial z_2} \right) \chi_{\{u_1^+ > u_2^+\}} (\varkappa' \varsigma_\lambda \widetilde{\varsigma}_\lambda)$$

which can be split as:

$$\begin{aligned}
&\int_{\Upsilon} h^3(z_1+t_1) \frac{\partial(u_1^+ - u_2^+)}{\partial z_2} \chi_{\{u_1^+ > u_2^+\}} \big(\varkappa'(z_2)\varsigma_\lambda(t_2)\widetilde{\varsigma}_\lambda(t_1)\big) \\
&\qquad + \int_{\Upsilon} \big[h^3(z_1+t_1) - h^3(z_1-t_1)\big] \frac{\partial u_2^+}{\partial z_2} \chi_{\{u_1^+ > u_2^+\}} (\varkappa' \varsigma_\lambda \widetilde{\varsigma}_\lambda) \\
&\quad = I_{1,1,1} + I_{1,1,2}.
\end{aligned}$$

The integral $I_{1,1,2}$ satisfies:

$$|I_{1,1,2}| \leqslant C \int_{\Upsilon} \big|h^3(z_1+t_1) - h^3(z_1-t_1)\big| \left| \frac{\partial u_2^+}{\partial z_2} \right| (\varsigma_\lambda \widetilde{\varsigma}_\lambda)$$

where constant C does not depend on parameter λ. By Hölder's inequality we obtain:

$$\begin{aligned}
|I_{1,1,2}| &\leqslant C \left\| \frac{\partial u_2^+}{\partial z_2} \right\| \left(\int_{\Upsilon} \big|h^3(z_1+t_1) - h^3(z_1-t_1)\big|^2 (\varsigma_\lambda \widetilde{\varsigma}_\lambda)^2 \right)^{\frac{1}{2}} \\
&\leqslant C' \sqrt{\lambda}
\end{aligned}$$

since h^3 is a Lipschitz continuous function. We conclude that

$$\lim_{\lambda\to 0} I_{1,1,2} = 0.$$

Moreover, by letting $\lambda \to 0$ (see [2]) in integral $I_{1,1,1}$, we get from (3.15)

$$\int_{\Omega} h^3(z_1+t_1) \frac{\partial(u_1^+(z+t) - u_2^+(z-t))^+}{\partial z_2} \varkappa'(z_2) \leqslant 0. \tag{3.16}$$

From this result one can demonstrate the following corollary.

Corollary 3.2. *If we have $\theta_0^1 \leqslant \theta_0^2$, then*

$$\int_{\Omega} h^3(z_1+t_1)\big(u_1^+(z+t) - u_2^+(z-t)\big)^+ \varkappa''(z_2) \geqslant 0, \quad \forall \varkappa \in C_c^{\infty,+}(\mathbb{R}).$$

Proof. The proof is obvious using relation (3.16) and Green's formula. □

Under the hypothesis that the input flow θ_0 is a Lipschitz function, we have the following monotonicity result:

Proposition 3.3. *The solution u^+ is monotone with respect to θ_0.*

Proof. Let us consider

$$D(z_2) = \int_0^1 h^3(z_1+t_1)\big(u_1^+(z+t) - u_2^+(z-t)\big)^+ \, dz_1$$

whose second derivative in the sense of distributions satisfies:

$$\big\langle D(z_2), \zeta \big\rangle = \int_\Omega h^3(z_1+t_1)\big(u_1^+(z+t) - u_2^+(z-t)\big)^+ \zeta''(z_2)\, dz_1\, dz_2 \geqslant 0,$$
$$\forall \zeta \in C_c^{\infty,+}(\mathbb{R}),$$

that is $\frac{d^2 D}{dx_2^2} \geqslant 0$ in $\mathcal{D}'(\Omega)$ and $D(0) = D(1) = 0$, and while applying the maximum principle [16], we find $D(z_2) \leqslant 0$, so that $\int_0^1 h^3(z_1+t_1)(u_1^+(z+t) - u_2^+(z-t))^+ \, dz_1$ is non-positive for every t in Ω. Then $u_1^+ \leqslant u_2^+$ in Ω and the proof is completed. $\square$

Using the same hypothesis (θ_0 is a Lipschitz function), we get the following uniqueness result:

Theorem 3.4. *The solution of the problem* $(\mathcal{P})$ *is unique.*

Proof. It is an immediate consequence of the previous proposition that if $\theta_0^1 = \theta_0^2$, then $u_1^+ = u_2^+$.
It follows from (3.2) that

$$\int_\Omega h(k_1 - k_2)\frac{\partial \phi}{\partial x}\, dx\, dy = 0 \quad \forall \phi \in V.$$

Let us remark that for every function ψ in $\mathcal{D}(\Omega)$ there exists ϕ in V such that $\frac{\partial \phi}{\partial x} = \psi$; we deduce that $k_1 = k_2$. $\square$

4. Remarks about the mechanical model and behaviour with respect to $\boldsymbol{\beta}$

4.1. Remarks about mechanical model

The idea of introducing equation like (1.2) to modelize the cavitation phenomena in lubrication area dates back to the work of Elrod [12] and was primarily devoted to provide a numerical approach for this phenomenon. The starting point is the usual Floberg cavitation equation with two unknowns, the pressure p and the saturation θ such that

$$\frac{\partial}{\partial x}\left(\frac{h^3}{12\eta}\frac{\partial p}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{h^3}{12\eta}\frac{\partial p}{\partial y}\right) = \frac{\partial}{\partial x}\left(\frac{h\theta U}{2}\right) \quad \text{with } p \geqslant P_{\text{cav}},$$
$$\theta \in H(p - P_{\text{cav}}), \tag{4.1}$$

H being the Heaviside graph and P_{cav} the vaporization pressure. This equation described a free boundary problems in which there exist two areas: one, the full film area where the pressure is greater than P_{cav} and the saturation is one while in the cavitation area, pressure equals P_{cav} and $0 \leqslant \theta \leqslant 1$. It has been widely studied both from existence and uniqueness [2,5,8]. Elrod remarks some analogy between the right-hand side of (4.1) and the one of the compressible Reynolds equation in witch ρ is the normalized density of the fluid [14]:

$$\frac{\partial}{\partial x}\left(\frac{\rho h^3}{12\eta}\frac{\partial p}{\partial x}\right)+\frac{\partial}{\partial y}\left(\frac{\rho h^3}{12\eta}\frac{\partial p}{\partial y}\right)=\frac{\partial}{\partial x}\left(\frac{\rho hU}{2}\right). \tag{4.2}$$

Introducing a specific compressibility law in the full film region

$$p=F(\rho)=P_{\mathrm{cav}}+\beta\log(\rho) \tag{4.3}$$

where β is a compressibility parameter and putting this law into (4.2) allows us to write:

$$\frac{\partial}{\partial x}\left(\frac{h^3}{12\eta}\frac{\partial \rho}{\partial x}\right)+\frac{\partial}{\partial y}\left(\frac{h^3}{12\eta}\frac{\partial \rho}{\partial y}\right)=\frac{\partial}{\partial x}\left(\frac{\rho hU}{2\beta}\right). \tag{4.4}$$

Retaining Eq. (4.4) when ρ is greater than one (or equivalently as $p>P_{\mathrm{cav}}$) and the equation

$$\frac{\partial}{\partial x}\left(\frac{\theta hU}{2}\right)=0$$

which is deduced from (4.1) in the cavitation area where $p=P_{\mathrm{cav}}$, it can be proposed a unique equation valid in both domains as:

$$\frac{\partial}{\partial x}\left(\frac{h^3}{12\eta}\frac{\partial(\theta-1)^+}{\partial x}\right)+\frac{\partial}{\partial y}\left(\frac{h^3}{12\eta}\frac{\partial(\theta-1)^+}{\partial y}\right)=\frac{\partial}{\partial x}\left(\frac{\theta hU}{2\beta}\right). \tag{4.5}$$

The solution θ of this problem, which is supposed to be easier to compute than the original one, will be identified to the saturation in Eq. (4.1) if $\theta\leqslant 1$ and to the density in Eq. (4.4) if $\theta>1$. In that last case, the pressure is recovered from Eq. (4.3).

The law (4.3) is a very specific one and the previous approach can be generalized to take into account more general compressibility laws.

Let us denote by $F_1(r)$ a strictly increasing regular function defined for $r\geqslant 1$ such that $F_1(1)=P_{\mathrm{cav}}$.

Performing now the change of unknown: $u_1=A_1(\rho)$ where $A_1(r)=\int_1^r tF_1'(t)\,dt$ and $B_1=A_1^{-1}$, the Reynolds equation in the non-cavitated area is written down:

$$\frac{\partial}{\partial x}\left(\frac{h^3}{12\eta}\frac{\partial u_1}{\partial x}\right)+\frac{\partial}{\partial y}\left(\frac{h^3}{12\eta}\frac{\partial u_1}{\partial y}\right)=\frac{\partial}{\partial x}\left(\frac{B_1(u_1)hU}{2}\right). \tag{4.6}$$

In the cavitation area, another pressure–density relation is introduced allowing the pressure to fall below the vaporization pressure $p=F_2(\rho)$ in which F_2 is a known, increasing function such that $F_2(1)=P_{\mathrm{cav}}$ and $F_1'(1)=F_2'(1)$ (for example see [10]).

Let us introduce now $u_2=A_2(\rho)$ where $A_2(r)=\int_1^r tF_2'(t)\,dt$ and $B_2=A_2^{-1}$. Pointing out that $u_1>0$ (respectively $u_2<0$) as soon as $p>P_{\mathrm{cav}}$ (respectively $p<P_{\mathrm{cav}}$), and defining

$$u=\begin{cases}u_1 & \text{in the non-cavitated area,}\\ u_2 & \text{in the cavitated area,}\end{cases}$$

and defining B by

$$B(u)=\begin{cases}B_1(u) & \text{if } u>0,\\ B_2(u) & \text{if } u<0,\end{cases}$$

then Eq. (4.1) can be rewritten as

$$\frac{\partial}{\partial x}\left(\frac{h^3}{12\eta}\frac{\partial u^+}{\partial x}\right)+\frac{\partial}{\partial y}\left(\frac{h^3}{12\eta}\frac{\partial u^+}{\partial y}\right)=\frac{\partial}{\partial x}\left(\frac{B(u)hU}{2}\right), \tag{4.7}$$

which is exactly the problem $(\mathcal{P})$ whose existence and uniqueness has been proved as soon as (1.3) is fulfilled.

4.2. Asymptotic case $\beta \to +\infty$

In the physical case described in the previous section, the parameter β in (1.3) appears to have physical meaning as a compressibility parameter. When this parameter tends to infinity, the function (F_1, F_2) tends to became the Heaviside graph and we get the following theorem.

Theorem 4.1. *As β tends to infinity, the positive part of the solution of problem tends to the solution of the incompressible* (*pressure–saturation*) *Reynolds equation* (4.1).

Proof. Starting point is the weak formulation of the problem (4.7) which is written down using the parameter β:

$$(\mathcal{P}_\beta)\quad \begin{cases} \text{Find } u_\beta \in L^2(\Omega) \text{ verifying } u_\beta^+ \in V,\ B(u_\beta) \geqslant 0, \\ \displaystyle\int_\Omega h^3 \nabla u_\beta^+ \nabla\phi\, dx\, dy - \int_\Omega h B(u_\beta) \frac{\partial \phi}{\partial x}\, dx\, dy = \int_{\Gamma_0} \theta_0 h \phi\, dy \quad \forall \phi \in V, \\ \theta_0 \in L^\infty(\Gamma_0),\ 0 \leqslant \theta_0 \leqslant 1. \end{cases}$$

In this section we will study the behavior of $(\mathcal{P}_\beta)$ as β tends to infinity. According to the previous Section 4.1 and taking Lemma 2.2 into account, it is convenient to introduce v_β and γ_β such that: $u_\beta = v_\beta + B^{-1}(\gamma_\beta)$ with $0 \leqslant \gamma_\beta \leqslant 1$. We can rewrite the previous equation:

$$\int_\Omega h^3 \nabla v_\beta \nabla \phi\, dx\, dy - \int_\Omega h\big(\gamma_\beta - 1 + B\big(v_\beta^+\big)\big) \frac{\partial \phi}{\partial x}\, dx\, dy = \int_{\Gamma_0} \theta_0 h \phi\, dy \quad \forall \phi \in V. \tag{4.8}$$

Taking in (4.8) $\phi = v_\beta$ in V, and from (1.3), we get $|B(u) - 1| \leqslant \frac{1}{\beta}|u|$; then we obtain the estimate

$$\min_{\{x\in[0,1]\}} \big(h^3(x)\big) \|v_\beta\|_V^2 \leqslant \|h\|_\infty \|\gamma_\beta\|_{L^2(\Omega)} \|v_\beta\|_V + \frac{1}{\beta} C_p \|h\|_{L^\infty(\Omega)} \|v_\beta\|_V^2 \\ + \|\theta_0\|_{L^2(\Omega)} \|h\|_{L^\infty(\Omega)},$$

i.e.,

$$\left(\min_{\{x\in[0,1]\}} \big(h^3(x)\big) - \frac{1}{\beta} C_p \|h\|_{L^\infty(\Omega)}\right) \|v_\beta\|_V^2 \\ \leqslant \|h\|_\infty \|\gamma_\beta\|_{L^2(\Omega)} \|v_\beta\|_V + \|\theta_0\|_{L^2(\Omega)} \|h\|_{L^\infty(\Omega)}.$$

From the condition $\min_{\{x\in[0,1]\}}(h^3(x)) - \frac{1}{\beta} C_p \|h\|_{L^\infty(\Omega)} > 0$, there exists α_0 such that

$$\frac{1}{\beta} < \alpha_0 \leqslant \frac{\min_{\{x\in[0,1]\}}(h^3(x))}{C_p \|h\|_{L^\infty(\Omega)}};$$

this implies $\|v_\beta\|_V^2 \leqslant C$. Then v_β weakly converges to a function p in $H^1(\Omega)$ and in addition γ_β *-weakly converges in $L^\infty(\Omega)$ towards θ with $0 \leqslant \theta \leqslant 1$. Passing to the limit in Eq. (4.8), we get

$$\int_\Omega h^3 \nabla p \nabla \phi \, dx\, dy - \int_\Omega h\theta \frac{\partial \phi}{\partial x}\, dx\, dy = \int_{\Gamma_0} \theta_0 h \phi \, dy \quad \forall \phi \in V. \tag{4.9}$$

Moreover, $p \geqslant P_{\text{cav}}$ in Ω. In addition $\gamma_\beta = 1$ as soon as $v_\beta > 0$ so that $\theta = 1$ when $p > P_{\text{cav}}$.

The application trace is continuous, therefore

$$p_{/\Gamma} = \lim_{\beta \to +\infty} u_{\beta/\Gamma} = 0.$$

Equation (4.9) is nothing else than a weak formulation of Eq. (4.1). $\square$

4.3. *Numerical aspect*

In the following, we will illustrate the convergence result obtained in the previous paragraph for the one dimension problem.

We consider the following equation corresponding to the case $B(u) = 1 + \frac{1}{\beta} u$, $h(x) = 1 + 0.5\cos(2\pi x)$ and $P_{\text{cav}} = 0$:

$$\frac{\partial}{\partial x}\left(h^3 g(u_\beta) \frac{\partial u_\beta}{\partial x} \right) = \frac{\partial (h(1 + \frac{1}{\beta} u_\beta))}{\partial x}, \quad 0 \leqslant x \leqslant 1 \tag{4.10}$$

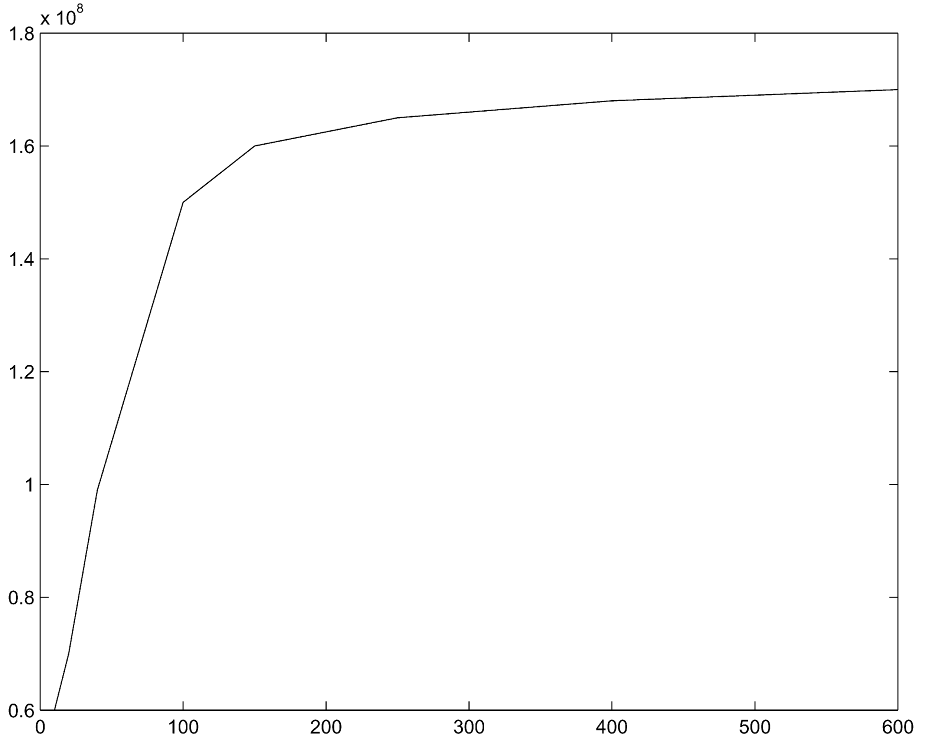

Fig. 2. $\|p_\beta\|_\infty$ versus β.

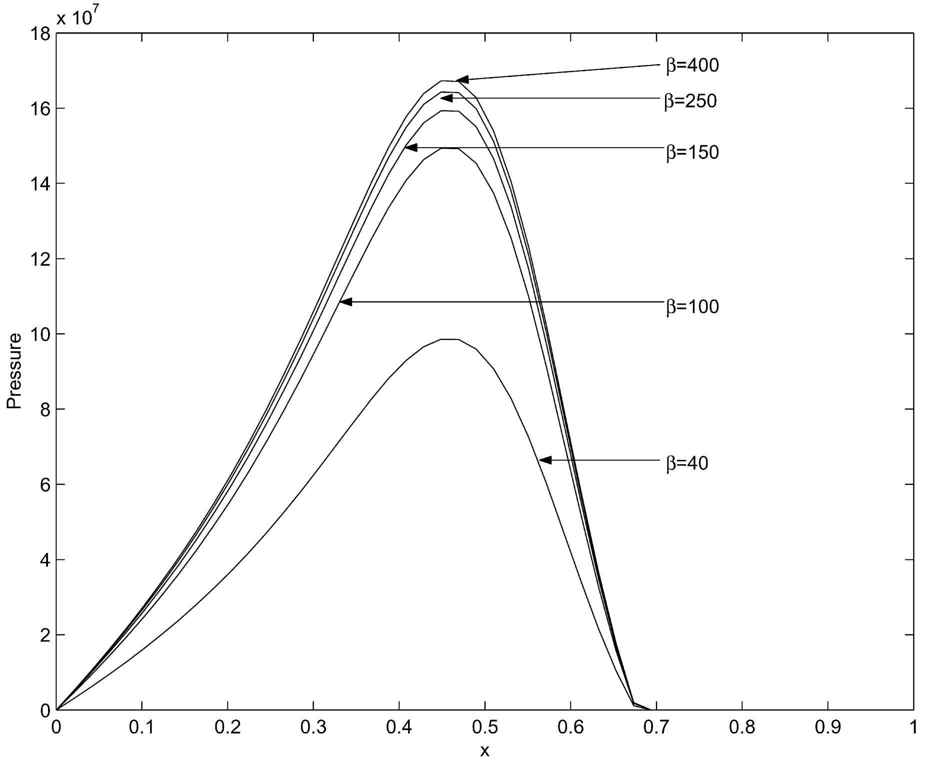

Fig. 3. $p_\beta(x)$ for different values of β.

with

$$g(u_\beta) = \begin{cases} 1 & \text{if } u_\beta \geqslant 0, \\ 0 & \text{if } u_\beta < 0. \end{cases}$$

A finite elements discretization for Eq. (4.10) can be written as follows choosing $z_\beta = 1 + \frac{1}{\beta} u_\beta$ as unknown whose components will be denoted by z_β^i:

$$a_i(z_\beta) z_\beta^{i-1} + b_i(z_\beta) z_\beta^i + c_i(z_\beta) z_\beta^{i+1} = d_i, \quad i = 2, \dots, N-1, \tag{4.11}$$

with

$$a_i(z_\beta) = \frac{-\beta}{\Delta x}\left(\frac{g_i h_i^3 + g_{i-1} h_{i-1}^3}{2}\right) - \frac{h_{i-1}}{2},$$

$$b_i(z_\beta) = \frac{\beta}{\Delta x}\left(\frac{g_{i+1} h_{i+1}^3 + g_{i-1} h_{i-1}^3}{2}\right),$$

$$c_i(z_\beta) = \frac{-\beta}{\Delta x}\left(\frac{g_i h_i^3 + g_{i+1} h_{i+1}^3}{2}\right) + \frac{h_{i+1}}{2}.$$

Note the dependance of the coefficient with respect of the unknown through the function

$$g_i(z_\beta) = \begin{cases} 1 & \text{if } z_\beta^i \geqslant 1, \\ 0 & \text{if } z_\beta^i < 1. \end{cases}$$

The boundary conditions will appear on the second member d_i. The non-linear system (4.11) can be also written as

$$A(z_\beta)z_\beta = D(z_\beta) \tag{4.12}$$

where A is the tridiagonal matrix (a_i, b_i, c_i) and the vector $D = (d_i)$. System (4.12) is then resolved by the fixed point iterative algorithm [13]:

$$A\big(z_\beta^{n+1}\big)z_\beta^n = D\big(z_\beta^n\big).$$

Initially all non-boundary nodes are assumed to be in the full film region, i.e., $z_\beta \geqslant 1$ and $g = 1$. Figure 2 depicts the variation of $\|p_\beta\|_\infty$ where $p_\beta = \beta \ln(1 + \frac{1}{\beta} u_\beta^+)$ according to β. The convergence towards an asymptotic value is obvious as β is great enough. In Fig. 3, we show p_β for different values of β.

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